QUANTIZATION OF THE KALUZA-KLEIN MONOPOLE SYSTEM BY PATH INTEGRATION

A. INOMATA
Department of Physics, State University of New York at Albany, 1400 Washington Avenue, Albany, NY 12222, USA

and

G. JUNKER
Physikalisches Institut der Universität Würzburg, Am Hubland, D-8700 Würzburg, FRG

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The Kaluza-Klein monopole system is quantized by path integration, and the exact energy spectrum is obtained. The radial Green function and radial wave functions are also found in closed form.

The Kaluza-Klein monopole of Gross and Perry [1] and of Sorkin [2] is a static solution of the classical field equation $R_{AB}=0$ in five-dimensional Kaluza-Klein theory, which is identical with the Taub-NUT instanton solution,

$$ds^2=V^{-1}[dr^2+r^2\,d\theta^2+r^2\sin^2\theta\,d\varphi^2]+V[4m\,d\psi+A_\varphi\,d\varphi]^2\,, \tag{1}$$

where (r,θ,φ) are polar coordinates, $x^5=4m\psi$, and

$$V=[1+4m/r]^{-1},\quad A_\varphi=4m(1-\cos\theta)\,. \tag{2}$$

The Kaluza-Klein monopole system, a test particle bound in the Taub-NUT space (1), has been quantized via Schrödinger's equations [3,4] and also by the supersymmetric WKB calculation [4] to find the exact energy spectrum. In this paper, we present a path integral quantization of the Kaluza-Klein monopole system, including the exact results for the energy spectrum and the radial wave functions.

In quantizing the Kaluza-Klein monopole system, Cordani, Fehér and Horvárthy [4] have derived the three-dimensional hamiltonian for a test particle of mass $M=1$ in the static field (1),

$$H=\frac{V}{2M}\boldsymbol{p}^2+\frac{M}{2V}q^2\,, \tag{3}$$

where q is the conserved charge,

$$q=4mV[\dot\psi+(1-\cos\theta)\dot\varphi]\,. \tag{4}$$

The angular part can be separated on the basis of monopole harmonics, so that the effective radial hamiltonian may be given by

$$H_r=\frac{V}{2M}p_r^2+\frac{l(l+1)-(4mq)^2}{2Mr^2}V+\frac{Mq^2}{2V}\,. \tag{5}$$

We utilize this result and write the corresponding lagrangian:

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$$L=\tfrac{1}{2}MV^{-1}\dot{r}^2-\frac{l(l+1)-(4mq)^2}{2Mr^2}V-\frac{Mq^2}{2V} \tag{6}$$

for which we attempt to carry out path integral quantization.

Due to the nontrivial structure of the background space, Feynman's path integral for the propagator cannot readily be computed. However, various path integral techniques newly developed have enabled us to find path integral solutions for systems such as the Kepler problem on uniformly curved spaces [5,6]. We shall apply similar techniques to the present problem. Instead of the propagator, we deal with the path integral for the promotor

$$P(\boldsymbol{r}'',\boldsymbol{r}';\tau)=\int \mathscr{D}[\boldsymbol{r}(t)]\,\exp\left(\frac{\mathrm{i}W[\boldsymbol{r}(t)]}{\hbar}\right), \tag{7}$$

where $W=S+E\tau$ is Hamilton's characteristic function. From this promotor, the energy dependent Green function, $G(\boldsymbol{r}'',\boldsymbol{r}';E)=\langle\boldsymbol{r}''|(E-H)^{-1}|\boldsymbol{r}'\rangle$, can be obtained by integration:

$$G(\boldsymbol{r}'',\boldsymbol{r}';E)=\frac{1}{\mathrm{i}\hbar}\int P(\boldsymbol{r}'',\boldsymbol{r}';\tau)\,\mathrm{d}\tau\,. \tag{8}$$

If we expand the Green function in terms of monopole harmonics as

$$G(\boldsymbol{r}'',\boldsymbol{r}';E)=\sum_{l=|q/\hbar|}^{\infty}G_l(r'',r';E)\sum_{\nu=-l}^{l}Y_{q/\hbar,l,\nu}(\theta'',\varphi'')\,Y^{*}_{q/\hbar,l,\nu}(\theta',\varphi') \tag{9}$$

then the radial Green function for the system (6) is given by

$$G_l(r'',r';E)=\frac{1}{\mathrm{i}\hbar}\int_0^{\infty}P_l(r'',r';\tau)\,\mathrm{d}\tau\,, \tag{10}$$

where

$$P_l(r'',r';\tau)=\frac{1}{r''r'}[V(r')V(r'')]^{1/2}\lim_{N\to\infty}\int\prod_{j=1}^{N}\exp\left(\frac{\mathrm{i}W_j^l}{\hbar}\right)\prod_{j=1}^{N}\left(\frac{M}{2\pi\mathrm{i}\hbar\tau_j}\right)^{1/2}\prod_{j=1}^{N-1}\sqrt{1+\frac{4m}{r_j}}\,\mathrm{d}r_j\,, \tag{11}$$

with

$$W_j^l=\frac{M}{2\tau_j}\hat{V}_j^{-1}(\Delta r_j)^2-\frac{l(l+1)\hbar^2-(4mq)^2}{2Mr_jr_{j-1}}\hat{V}_j\tau_j-\frac{Mq^2}{2\hat{V}_j}\tau_j+E\tau_j\,. \tag{12}$$

In the above, we have used the notations: $t''=t_N$, $t'=t_0$, $r_j=r(t_j)$, $\Delta r_j=r_j-r_{j-1}$, $\tau_j=t_j-t_{j-1}$, $\tau=\sum_{j=1}^{N}\tau_j$, $\hat{r}_j^2=r_jr_{j-1}$, and $\hat{V}_j^2=[(1+4m/r_j)(1+4m/r_{j-1})]^{-1}$. As in refs. [7,8], we have chosen in evaluating the short time action (12) the geometric mean value prescription which corresponds to a symmetric operator ordering, $f(r)p^2f(r)$.

Now we apply the local space–time transformation:

$$\rho_j=\sqrt{r_j},\quad \sigma_j=\tau_j\left[4\rho_j\rho_{j-1}\sqrt{\left(1+\frac{4m}{\rho_j^2}\right)\left(1+\frac{4m}{\rho_{j-1}^2}\right)}\right]^{-1}, \tag{13}$$

with the global scaling

$$\tau=4\sigma\rho''\rho'\sqrt{\left(1+\frac{4m}{\rho''^2}\right)\left(1+\frac{4m}{\rho'^2}\right)}, \tag{14}$$

where $\sigma=\sum_{j=1}^{N}\sigma_j$. This local transformation takes the promotor (11) into another promotor which is equivalent to the former when integrated over τ.

Expansion of the kinetic term in (12) up to $O(\Delta\rho^4)$ yields

$$\frac{M}{2\tau_j}\hat{V}_j^{-1}(\Delta r_j)^2 \doteq \frac{M}{2\sigma_j}\left((\Delta\rho_j)^2+\frac{(\Delta\rho_j)^4}{4\rho_j\rho_{j-1}}\right) \doteq \frac{M}{2\sigma_j}(\Delta\rho_j)^2-\frac{3\hbar^2\sigma_j}{8M\rho_j\rho_{j-1}}.$$

The notation $\doteq$ implies equivalence in the path integral via the McLaughlin–Schulman procedure. The measure also changes as

$$\prod_{j=1}^{N}\left(\frac{M}{2\pi i\hbar\tau_j}\right)^{1/2}\prod_{j=1}^{N-1}\sqrt{1+\frac{4m}{r_j}}\,dr_j=(4\rho''\rho')^{-1/2}\left[\left(1+\frac{4m}{\rho'^2}\right)\left(1+\frac{4m}{\rho''^2}\right)\right]^{-1/4}\prod_{j=1}^{N}\left(\frac{M}{2\pi i\hbar\sigma_j}\right)^{1/2}\prod_{j=1}^{N-1}d\rho_j .$$

Thus the path integral (11) can be written as

$$P_l(r'',r';\tau)=\tfrac{1}{2}(\rho''\rho')^{-3/2}\left[\left(1+\frac{4m}{\rho'^2}\right)\left(1+\frac{4m}{\rho''^2}\right)\right]^{-3/4}\exp\left(\frac{i16m\sigma(E-Mq^2)}{\hbar}\right)\tilde{K}_\lambda(\rho'',\rho';\sigma), \tag{15}$$

where

$$\tilde{K}_\lambda(\rho'',\rho';\sigma)=\frac{1}{\rho''\rho'}\lim_{N\to\infty}\int\prod_{j=1}^{N}\left(\frac{M}{2\pi i\hbar\sigma_j}\right)^{1/2}\exp\left(\frac{i\tilde{S}(\sigma_j)}{\hbar}\right)\prod_{j=1}^{N-1}d\rho_j,$$

$$\tilde{S}(\sigma_j)=\frac{M}{2\sigma_j}(\Delta\rho_j)^2-\frac{(2l+1)^2-\frac{1}{4}}{2M\rho_j\rho_{j-1}}\hbar^2\sigma_j+(4E-2q^2M)\rho_j\rho_{j-1}\sigma_j . \tag{16}$$

Setting

$$\lambda=2l+\tfrac{1}{2},\quad \omega^2=-\frac{8(E-\frac{1}{2}q^2M)}{M},\quad e^2=4m(E-Mq^2), \tag{17}$$

we see that the path integral (16) is identical in form with the radial path integral of a harmonic oscillator of angular momentum λ. It has been evaluated for the radial path integration of the hydrogen atom [8], the result being

$$\tilde{K}_\lambda(\rho'',\rho';\sigma)=(\rho'\rho'')^{-1/2}\frac{M\omega}{i\hbar}\csc(\omega\sigma)\exp\left(\frac{iM\omega}{2\hbar}(\rho'^2+\rho''^2)\cot(\omega\sigma)\right)I_{\lambda+1/2}\left(\frac{M\omega}{i\hbar}\rho'\rho''\csc(\omega\sigma)\right), \tag{18}$$

where $I_\nu(z)$ is the modified Bessel function. Substituting this into (15) and integrating the promotor with the help of the integral formula

$$\int_0^\infty \exp(-2\alpha p)\exp[-\tfrac{1}{2}(x+y)\coth\alpha]\,I_{2\nu}[(xy)^{1/2}\operatorname{csch}\alpha]\operatorname{csch}\alpha\,d\alpha$$

$$=\left(\frac{\Gamma(p+\nu+\frac{1}{2})}{(xy)^{1/2}\Gamma(2\nu+1)}\right)M_{-p,\nu}(x)W_{-p,\nu}(y), \tag{19}$$

we find the radial Green function in closed form,

$$G_l(r'',r';E)=\frac{2M}{\hbar^2}\left[\left(1+\frac{4m}{r''}\right)\left(1+\frac{4m}{r'}\right)\right]^{-1/4}(2ikr''r')^{-1}\frac{\Gamma(p+l+1)}{\Gamma(2l+2)}$$

$$\times M_{-p,l+1/2}(-2ikr')W_{-p,l+1/2}(-2ikr''), \tag{20}$$

where $k=M\omega/2i\hbar$, $p=-iMe^2/\hbar^2k$, and $M_{\alpha,\beta}(x)$ and $W_{\alpha,\beta}(x)$ are the Whittaker functions. From the poles of (20) at $p=-n_r-l-1$ we deduce the energy spectrum

$$E_n=\frac{\hbar^2}{(4m)^2M}(n^2-s^2)^{1/2}[\pm n-(n^2-s^2)^{1/2}],\quad n=n_r+l+1,\, n_r=0,1,2,\ldots,\, l\geqslant s\,, \tag{21}$$

where $s=4mMq/\hbar$. This coincides with the result obtained by Gibbons and Manton [3] and by Gordani, Fehér and Horváthy [4]. Note that the poles arise only for positive ω^2 and e^2 from which follow the conditions $E<\frac{1}{2}q^2M$ and $m<0$. The upper sign of (21) corresponds to positive energy eigenvalues, whereas the lower sign leads to $E_n<0$. A more detailed discussion of the spectrum (21) can be found in ref. [3].

The residues of (20) determine the normalized wave functions in terms of Laguerre polynomials:

$$R_{nl}(r)=\left(1+\frac{4m}{r}\right)^{-1/4}\left(\frac{4n_r!}{a^3n^4\Gamma(n_r+2l+2)}\right)^{1/2}\left(\frac{2r}{an}\right)^l\exp(-r/an)L_{n_r}^{(2l+1)}\left(\frac{2r}{an}\right), \tag{22}$$

where $a=\hbar^2(q^2M^2-2EM)^{1/2}/n$. These functions are normalized in L^2 space with the measure $(1+4m/r)^{1/2}\,dr$.

In this article, we have started with the lagrangian corresponding to the hamiltonian obtained by Cordani, Fehér and Horváthy to find in a short cut an exact energy spectrum and the radial wave functions in closed form. Although the conserved charge q has been fixed to a single constant value, the charge quantization can also be accomodated within the path integral scheme. This and an alternative account of the path integral treatment of the Kaluza–Klein monopole system based on the Kustaanheimo–Stiefel coordinates will be given elsewhere.

After completion of this manuscript, the present authors have become aware of Bernido's article [9] where our alternative approach [10] is utilized.

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